

INVARIANTS OF SETS OF POINTS UNDER INVERSION

Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS
UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

JOSEPH WILLIAM PETERS

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Invariants of Sets of Points under Inversion.

By JOSEPH WILLIAM PETERS.

1. *Introduction.* Two points, P and P' , on a line with a point O are called inverse points if $OP \cdot OP' = \kappa$, where κ is a real constant. The point O is the center of the inversion. The transformation applies not only to points in a plane but to points in flat spaces of any number dimensions. For $\kappa > 0$, there is in the plane a circle of fixed points, in three dimensions a sphere of fixed points, and so on. To every point P different from O , there corresponds a unique finite point. In order to have a one-to-one correspondence in all cases, we close our space by a single point at infinity which is the inverse of the center, O . As a consequence of this assumption, circles and lines in the plane are inversely equivalent; spheres and planes in three dimensions are inversely equivalent; and so on. A further important property of inversions concerns the relationship between the distance from P to Q and the distance between the inverse points P' and Q' . If O is the center of inversion and κ the constant of inversion, the relation is

$$PQ = (\kappa/OP' \cdot OQ') \cdot P'Q'.$$

With these general remarks concerning inversions in mind, let us now proceed to the discussion of invariants under inversion.

2. *Inversive Properties of Sets of Points.* We shall first take the case of three points. Let (ij) represent the distance between the points i and j and let $(ij) = (ji)$. If O be a variable point in the plane of the three fixed points 1, 2, 3, consider the equations

$$(2.1) \quad (01)(23) = (02)(31) = (03)(12).$$

Under an inversion with any other point in the plane of 1, 2, 3 as a center, these equations are reproduced in the primed numbers. Hence we will say that they are invariant under inversions, or simply inversive.

The equations (2.1) represent three circles which intersect in two real points. For if we let the point 3 approach infinity, the equations of the three circles become

$$(01) = (12) \quad (02) = (12) \quad (01) = (02).$$

The first circle has its center at 1 and radius (12) , the second has its center

at 2 and radius (12), hence they intersect on the perpendicular bisector of (12) which is the third circle.

Since the circles (2.1) intersect in two real points, let us move one of these points to infinity by using it as the center of an inversion. The circles (2.1) now become straight lines with the equations

$$(01) = (02) = (03),$$

and the triangle 1, 2, 3 becomes equilateral since

$$(12) = (23) = (31).$$

It is now evident that each circle passes through one vertex of the triangle 1, 2, 3 and has the other two vertices as inverse points. Hence the equations $(01) = (02) = (03)$ represent the Apollonian Circles of the equilateral triangle 1, 2, 3.

In general, the equations (2.1) represent the Apollonian Circles of the triangle 1, 2, 3. Their two points of intersection are called the Hessian points of the triangle.

Consider now the equations

$$(2.2) \quad (01)(23) \pm (02)(13) \pm (03)(12) = 0$$

using all combinations of signs. The equations are inversive. If we invert with one of the Hessian points as a center, the triangle 1, 2, 3 becomes equilateral as before, and the equations (2.2) become

$$(01) \pm (02) \pm (03) = 0.$$

The case where the signs are all positive is an inversion in the circumcircle of the triangle 1, 2, 3.

$(01) + (02) - (03) = 0$ represents the arc of the circumcircle from the point 1 to the point 2.

$(01) - (02) + (03) = 0$ represents the arc of the circumcircle from the point 3 to the point 1.

$(01) - (02) - (03) = 0$ represents the arc of the circumcircle from the point 2 to the point 3.

The point simultaneously satisfying $(01) = (02) + (03)$ and $(02) = (03)$ is on the circumcircle and is called the counterpoint of 1. Similarly $(02) = (01) + (03)$ and $(01) = (03)$ give the counterpoint of 2 and $(03) = (01) + (02)$ and $(01) = (02)$ give the counterpoint of 3.

To summarize, we may say the invariant relations of three points under inversion give the Hessian points; the Apollonian Circles; the circumcircle of the three points, not as a whole but in three arcs which make it complete;

an inversion in the circumcircle; and lastly the counterpoints of the three given points.

Take now the case of four points. Let 1, 2, 3, 4 be four points in a three-dimensional space with 0 as a variable point in that space and consider the equations homogeneous in the points 0, 1, 2, 3, 4

$$(2.3) \quad \begin{aligned} (01)^2(23)(34)(42) &= (02)^2(34)(41)(13) \\ &= (03)^2(41)(12)(24) = (04)^2(12)(23)(31). \end{aligned}$$

These six equations are inversive. They represent six spheres which intersect in two points, for if the point 4 approaches infinity the equations (2.3) become

$$(01)^2(23) = (02)^2(13) = (03)^2(12) = (12)(23)(13).$$

The spheres $(01)^2 = (12)(13)$ and $(02)^2 = (12)(23)$ will intersect if

$$(01) + (02) > (12)$$

or

$$(13)^{\frac{1}{2}} + (23)^{\frac{1}{2}} > (12)^{\frac{1}{2}}.$$

We assume that none of the four points coincide, so $(13) + (23) > (12)$. This inequality implies $(13)^{\frac{1}{2}} + (23)^{\frac{1}{2}} > (12)^{\frac{1}{2}}$. A similar proof shows that $(01)^2 = (12)(13)$ and $(03)^2 = (23)(13)$ intersect and likewise $(02)^2 = (12)(13)$ and $(03)^2 = (23)(13)$ intersect. These three spheres will intersect in two points and from the original conditions imposed by the equations the other three spheres must also pass through these two points, called the canonizant pair.

Let us put one of these canonizant points at infinity by an inversion with that point as a center. Then the six spheres become six planes whose equations are given by

$$(01)^2 = (02)^2 = (03)^2 = (04)^2.$$

The distances between the four points will then satisfy the following equations:

$$(23)(34)(42) = (34)(41)(13) = (14)(12)(24) = (12)(23)(13),$$

which reduce to

$$(12)^2 = (34)^2, \quad (13)^2 = (24)^2, \quad (14)^2 = (23)^2.$$

Four points in a three-dimensional space are always on a sphere, and these equations arrange the four points so that, if the center of the sphere is the origin, the points may be given coordinates (a, b, c) , $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$. In this same reference frame, the Cartesian equations of the six planes are

$$\begin{array}{ll}
(01)^2 = (02)^2 & by + cz = 0, \\
(01)^2 = (03)^2 & ax + cz = 0, \\
(01)^2 = (04)^2 & ax + by = 0, \\
(02)^2 = (03)^2 & ax - by = 0, \\
(02)^2 = (04)^2 & ax - cz = 0, \\
(03)^2 = (04)^2 & by - cz = 0.
\end{array}$$

Thus when one of the canonizant points is at infinity, the other is at the center of the sphere on the four points.

Consider now the following equations, using all combinations of signs:

$$\begin{aligned}
(2.4) \quad & (01)^2(23)(34)(42) \pm (02)^2(34)(41)(13) \\
& \pm (03)^2(41)(12)(24) \pm (04)^2(12)(23)(31) = 0.
\end{aligned}$$

The eight equations are inversive and represent spheres or inversions. To study these equations, it will be convenient to put one of the canonizant points at infinity, as was done before. Then the equations (2.4) become

$$(01)^2 \pm (02)^2 \pm (03)^2 \pm (04)^2 = 0,$$

and the points 1, 2, 3, 4 may be given coordinates (a, b, c) , $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$, respectively. To determine the nature of these spheres or inversions, we will find their equations in the Cartesian coordinate system determined by the naming of the four points.

$$(01)^2 + (02)^2 + (03)^2 + (04)^2 = 0$$

becomes

$$x^2 + y^2 + z^2 = -(a^2 + b^2 + c^2).$$

This represents an inversion with $(0, 0, 0)$ as the center. The sphere of fixed points does not exist. Under this inversion

$$\begin{array}{ll}
(a, b, c) & \rightarrow (-a, -b, -c), \\
(a, -b, -c) & \rightarrow (-a, b, c), \\
(-a, b, -c) & \rightarrow (a, -b, c), \\
(-a, -b, c) & \rightarrow (a, b, -c).
\end{array}$$

The right-hand column gives four points known as the counterpoints of the first set. The equation

$$(01)^2 + (02)^2 + (03)^2 - (04)^2 = 0$$

becomes in the Cartesian reference frame

$$(x-a)^2 + (y-b)^2 + (z+c)^2 = 0.$$

It evidently represents the point $(a, b, -c)$. Likewise

$$(01)^2 + (02)^2 - (03)^2 + (04)^2 = 0$$

represents the point $(a, -b, c)$,

$$(01)^2 - (02)^2 + (03)^2 + (04)^2 = 0$$

represents the point $(-a, b, c)$,

and
$$(01)^2 - (02)^2 - (03)^2 - (04)^2 = 0$$

represents the point $(-a, -b, -c)$. Thus for an odd number of minus signs in the equations (2.4) the counterpoints of the four given points appear. The relation of the four points to the four counterpoints is mutual. The

sphere
$$(01)^2 + (02)^2 - (03)^2 - (04)^2 = 0$$

represents the coordinate plane $x=0$, in the Cartesian system chosen.

Likewise
$$(01)^2 - (02)^2 + (03)^2 - (04)^2 = 0$$

represents the coordinate plane $y=0$, and

$$(01)^2 - (02)^2 - (03)^2 + (04)^2 = 0$$

represents the coordinate plane $z=0$. These three spheres (or planes) are mutually perpendicular and intersect on the canonizant points, which in this case are $(0, 0, 0)$ and ∞ . The sphere containing the four points 1, 2, 3, 4 and the planes $x=0$ and $y=0$ intersect in the points $(0, 0, \pm 1)$ if the sphere is regarded as being of unit radius. Likewise the sphere and the planes $x=0$ and $z=0$ intersect at $(0, \pm 1, 0)$, and the sphere and the planes $y=0$ and $z=0$ intersect at $(\pm 1, 0, 0)$. These six points are called the Jacobian points of 1, 2, 3, 4 and the three spheres (or planes) $x=0$, $y=0$, $z=0$ are called the Jacobian spheres.

The inversive theory of four points gives, in short, six spheres which intersect in two points called the canonizant pair. It also gives the four counterpoints and an inversion connecting them with the original four points. Likewise the three Jacobian spheres and the six Jacobian points appear.

Since four points in three-dimensional space are always on a sphere, we may consider the special case when the sphere is a plane. The counterpoints will also be in the plane and the inversion sending them into the original

four points will be determined. Since the three Jacobian spheres intersect the sphere containing the four points, they will intersect the plane in three circles called the Jacobian circles. These circles will be perpendicular to each other and intersect in the six Jacobian points.

Let us now attempt to extend this theory to five points in a four-dimensional space. We will first consider the equations

$$(2.5) \quad \frac{(01)^3}{(12)(13)(14)(15)} = \frac{(02)^3}{(12)(23)(24)(25)} = \frac{(03)^3}{(13)(23)(34)(35)} \\ = \frac{(04)^3}{(14)(24)(34)(45)} = \frac{(05)^3}{(15)(25)(35)(45)}.$$

These equations are inversive and represent sixteen hyperspheres. However, if we try to prove that the spheres intersect as in the case of four points, we find that they may not. Hence, if they do not intersect, we do not have a canonizant pair of points as we had in the other cases. We have, however, a canonizant circle.

Let us now consider the equations

$$(2.6) \quad \frac{(01)^3}{(12)(13)(14)(15)} \pm \frac{(02)^3}{(12)(23)(24)(25)} \pm \frac{(03)^3}{(13)(23)(34)(35)} \\ \pm \frac{(04)^3}{(14)(24)(34)(45)} \pm \frac{(05)^3}{(15)(25)(35)(45)} = 0.$$

We have here sixteen equations representing hyperspheres or inversions. Let us suppose the five points so chosen that the equations (2.6) become

$$(01)^3 \pm (02)^3 \pm (03)^3 \pm (04)^3 \pm (05)^3 = 0.$$

The conditions imposed on the five points in that case are

$$(12)(13)(14)(15) = (12)(23)(24)(25) = (13)(23)(34)(35) \\ = (14)(24)(34)(45) = (15)(25)(35)(45).$$

They do not seem to place the points in any symmetrical arrangement.

If we calculate the "power" or bilinear invariant of each hypersphere of the set

$$(01)^3 \pm (02)^3 \pm (03)^3 \pm (04)^3 \pm (05)^3 = 0$$

with itself, we find there are six inversions and ten real hyperspheres. The six inversions are given by the one equation with all plus signs and the five equations with one minus sign. The ten hyperspheres are given by the ten

equations with two minus signs. The geometric connection between these hyperspheres and inversions and the five points is not evident, and in this paper we will leave the theory here.

3. *The Cayley Determinants.* Cayley* has shown that for four points to be on a circle, the following determinant is zero,

$$C_4 \equiv \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & 0 & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & 0 & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 0 \end{vmatrix} = 0$$

where λ_{ij} is the square of the distance between the points i and j and $\lambda_{ij} = \lambda_{ji}$. Five points on a sphere satisfy a similar relation, $C_5 = 0$, where C_5 is a five-rowed determinant similar to C_4 .

In the same paper, Cayley gave the condition for three points to be on a line, namely

$$D_3 \equiv \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & 1 \\ \lambda_{21} & 0 & \lambda_{23} & 1 \\ \lambda_{31} & \lambda_{32} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

Four points in a plane satisfy the relation $D_4 = 0$ where D_4 is the determinant C_4 with a bordering row and column of units. Similarly five points in three-dimensional space satisfy $D_5 = 0$ where D_5 is the determinant C_5 with a bordering row and column of units.

The relations $C_4 = 0$ and $C_5 = 0$ are inversive, for if we replace λ_{ij} by $\kappa^2 \lambda'_{ij} / \lambda'_{ci} \lambda'_{cj}$, where c is the center of the inversion and not one of the points, $C_4 = 0$ is reproduced in the primed letters, and likewise $C_5 = 0$ becomes $C'_5 = 0$. If the center of the inversion, c , is one of the given points, then $C_4 = 0$ becomes $D_3 = 0$, the fourth point having gone into the point at infinity. Likewise if we invert with the point 5 as a center, $C_5 = 0$ becomes $D_4 = 0$. The relations $D_3 = 0$, $D_4 = 0$, $D_5 = 0$ are not inversive. If we invert with a point c , not in the line of 1, 2, 3 as a center, the point c goes to infinity, the point at infinity becomes a point 4, and $D_3 = 0$ becomes $C_4 = 0$. Likewise we may derive the condition that five points be on a sphere, $C_5 = 0$, by an inversion applied to $D_4 = 0$, where the center of the inversion must not be in the plane of the four points satisfying $D_4 = 0$.

* Cayley, *Collected Mathematical Papers*, Vol. I, p. 1; also Salmon, *Conic Sections*, 6th ed. (1879), p. 134.

It is evident that determinants such as these may be set down for any number of points. $D_n = 0$ is the relation connecting any n points in S_{n-2} . $C_n = 0$ is the condition that n points be on a "round" or hypersphere in S_{n-2} . The relations $C_n = 0$ are all inversive.

The question arises as to what geometric interpretation we may give these determinants when they are not zero. The following two theorems will answer the question for three points and then extensions will be made to any number of points.

THEOREM 1. *If Δ be the area of the triangle formed by any three points, then*

$$D_3 = -16\Delta^2.$$

Proof: Three points on a line satisfy the relation $D_3 = 0$. If the point 3 is at a distance p from the line joining 1 and 2, then

$$\begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} - p^2 & 1 \\ \lambda_{21} & 0 & \lambda_{23} - p^2 & 1 \\ \lambda_{31} - p^2 & \lambda_{32} - p^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

On splitting this determinant up into sums, we find that $D_3 + 2p^2 D_2 = 0$. But $D_2 = 2\lambda_{12}$ and since $p^2\lambda_{12}$ is four times the square of the area of the triangle 1, 2, 3, then $D_3 = -16\Delta^2$.

THEOREM 2. *If r be the radius of the circumcircle of any three points not in a straight line, then $C_3 = -2r^2 D_3$.*

Proof: For three points not in a straight line, we have

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) \\ \cos(21) & 1 & \cos(23) \\ \cos(31) & \cos(32) & 1 \end{vmatrix} = 0$$

where $\cos(ij)$ is the cosine of the angle subtended at the circumcenter by the arc of the circumcircle joining the points i and j and $\cos(ij) = \cos(ji)$. If we replace $\cos(ij)$ by $1 - \lambda_{ij}/2r^2$, where r is the radius of the circumcircle and λ_{ij} is the square of the distance between the points i and j , the determinant becomes

$$\begin{vmatrix} 1 & 1 - \lambda_{12}/2r^2 & 1 - \lambda_{13}/2r^2 \\ 1 - \lambda_{21}/2r^2 & 1 & 1 - \lambda_{23}/2r^2 \\ 1 - \lambda_{31}/2r^2 & 1 - \lambda_{32}/2r^2 & 1 \end{vmatrix} = 0.$$

After splitting this determinant up into a sum of determinants, we find $C_3 + 2r^2 D_3 = 0$. Since from theorem 1, $D_3 = -16\Delta^2$, we have $C_3 = 32r^2\Delta^2$.

The following are the corresponding theorems for four points not in a plane, with a sketch of the proofs:

THEOREM 1': If V be the volume of the tetrahedron formed by the four points 1, 2, 3, 4, then $D_4 = 288V^2$.*

Proof: If p be the distance from the plane of the points 1, 2, 3 to the point 4, we may replace in the relation $D_4 = 0$, λ_{i4} by $\lambda_{i4} - p^2$ ($i = 1, 2, 3$). Splitting the resulting determinant up into sums, we find $D_4 + 2p^2 D_3 = 0$.

Since $D_3 = -16\Delta^2$, $D_4 = 32p^2\Delta^2$.

But since $p \Delta = 3V$, we have $D_4 = 288V^2$.

THEOREM 2': If r be the radius of the sphere on the four points, then

$$C_4 = -2r^2 D_4.$$

Proof: Four points in a three-dimensional space always satisfy the relation †

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) & \cos(14) \\ \cos(21) & 1 & \cos(23) & \cos(24) \\ \cos(31) & \cos(32) & 1 & \cos(34) \\ \cos(41) & \cos(42) & \cos(43) & 1 \end{vmatrix} = 0,$$

where (ij) is the arc of the great circle joining the points i and j on the sphere on the four points. Replacing $\cos(ij)$ in this determinant by $1 - \lambda_{ij}/2r^2$ and splitting up the resulting determinant into sums, we find

$$C_4 + 2r^2 D_4 = 0.$$

Since $D_4 = 288V^2$, we have

$$C_4 = -576r^2 V^2.$$

These theorems can be extended to any number of points. Thus we have

THEOREM I. If V be the content of the solid formed by n points in S_{n-1} , then $D_n = kV^2$, where $k = (-1)^n \cdot (2)^{n-1} [(n-1)!]^2$.

* Salmon, *Analytic Geometry of Three Dimensions*, 3d ed. (1874), p. 34.

† Salmon, *loc. cit.*, p. 35.

THEOREM II. If r be the radius of the hypersphere in S_{n-1} on which the n points lie, then $C_n = -2r^2 D_n$, or $C_n = 2kr^2 V^2$, where k is defined as above.

4. *The Doubly Bordered Cayley Determinant.* Four points in a plane satisfy the relation $D_4 = 0$. If about the point 4, a circle of radius ρ is drawn, then

$$\lambda_{14} = (\delta_1 + \rho)^2, \quad \lambda_{24} = (\delta_2 + \rho)^2, \quad \lambda_{34} = (\delta_3 + \rho)^2,$$

where $\delta_1, \delta_2, \delta_3$ represent perpendicular distances from the points 1, 2, 3, respectively, to the circumference of the circle. The relation, a quadratic in ρ , is now written

$$\begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & (\delta_1 + \rho)^2 & 1 \\ \lambda_{21} & 0 & \lambda_{23} & (\delta_2 + \rho)^2 & 1 \\ \lambda_{31} & \lambda_{32} & 0 & (\delta_3 + \rho)^2 & 1 \\ (\delta_1 + \rho)^2 & (\delta_2 + \rho)^2 & (\delta_3 + \rho)^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

If now the point 4 moves off to infinity, ρ becomes infinite, the circle becomes a straight line, and the coefficient of ρ^2 in the above relation is zero. Therefore

$$2 \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & \delta_1 & 1 \\ \lambda_{21} & 0 & \lambda_{23} & \delta_2 & 1 \\ \lambda_{31} & \lambda_{32} & 0 & \delta_3 & 1 \\ \delta_1 & \delta_2 & \delta_3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & 1 \\ \lambda_{21} & 0 & \lambda_{23} & 1 \\ \lambda_{31} & \lambda_{32} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

The determinant on the left is Cayley's C_3 doubly bordered, first with δ 's, then with units. The determinant on the right is D_3 which is equal to $-16\Delta^2$ where Δ is the area of the triangle formed by the three points 1, 2, 3. Let $\lambda_{23} = a^2$, $\lambda_{13} = b^2$, $\lambda_{12} = c^2$ and let A, B, C be the interior angles of the triangle 1, 2, 3 at the points 1, 2, 3 respectively. Then if the left-hand determinant is expanded, we have

$$(4.1) \quad a^2\delta_1^2 + b^2\delta_2^2 + c^2\delta_3^2 - 2ab \cos C \delta_1\delta_2 \\ - 2ac \cos B \delta_1\delta_3 - 2bc \cos A \delta_2\delta_3 = 4\Delta^2.$$

The $\delta_1, \delta_2, \delta_3$ now represent the perpendicular distances from the points 1, 2, 3 to a straight line. For a given δ_2 and δ_3 there are ordinarily two straight lines satisfying the equation. They are common tangents to the

circles with centers at 2 and 3 and radii δ_2 and δ_3 respectively. They are internal or external pairs according as 2 and 3 are on the opposite or the same side of the lines. The internal tangents coincide when $\delta_2 - \delta_3 = \pm a$ and likewise the external tangents coincide when $\delta_2 + \delta_3 = \pm a$. Algebraically we may therefore say that the discriminant of equation (4.1) with respect to δ_1 , when set equal to zero, must reduce to $\delta_2 - \delta_3 = \pm a$. The discriminant set equal to zero gives

$$b^2 \sin^2 C \delta_2^2 + c^2 \sin^2 B \delta_3^2 - 2bc (\cos A + \cos B \cos C) \delta_2 \delta_3 = 4\Delta^2.$$

This equation should reduce to $(\delta_2 - \delta_3)^2 = a^2$. Therefore the following relations must be satisfied.

$$\begin{aligned} b^2 \sin^2 C &= bc (\cos A + \cos B \cos C), \\ b/c &= \sin B / \sin C, \quad ab \sin C = 2\Delta. \end{aligned}$$

The first relation is a form of the law of cosines for a triangle. The second is the law of sines. The third gives an expression for the area. The first relation gives

$$b^2 \sin^2 C = bc \cos A + bc \cos (B + C) + bc \sin B \sin C,$$

which, upon using the second relation becomes $\cos A = -\cos (B + C)$, whence $A + B + C = \pi$. Here we have sufficient information about the triangle to determine any of its properties. Can we do the same for four points in S_3 ?

To find the properties of the tetrahedron, or four point in S_3 , start with the five points satisfying the relation $D_5 = 0$. About the point 5 draw a sphere of radius ρ , and let $\delta_1, \delta_2, \delta_3, \delta_4$ be perpendicular distances from the points 1, 2, 3, 4 to the sphere. Then in the relation $D_5 = 0$, set $\lambda_{i5} = (\delta_i + \rho)^2$, where $i = 1, 2, 3, 4$. If now the point 5 moves off to infinity in a given direction, ρ becomes infinite, the sphere becomes a plane, and the coefficient of ρ^2 in the determinant $D_5 = 0$ is zero. Therefore

$$\begin{vmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} & \delta_1 & 1 \\ \lambda_{21} & 0 & \lambda_{23} & \lambda_{24} & \delta_2 & 1 \\ \lambda_{31} & \lambda_{32} & 0 & \lambda_{34} & \delta_3 & 1 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 0 & \delta_4 & 1 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{vmatrix} = 144V^2,$$

where V is the volume of the tetrahedron 1, 2, 3, 4. The above determinant is C_4 , doubly bordered, first with δ 's, then with units. Remembering that $D_3 = -16\Delta^2$, we have on expanding the above determinant

$$\begin{aligned}
 (4.2) \quad & \Delta_1^2 \delta_1^2 + \Delta_2^2 \delta_2^2 + \Delta_3^2 \delta_3^2 + \Delta_4^2 \delta_4^2 \\
 & - 2\Delta_1 \Delta_2 \delta_1 \delta_2 \cos(12) - 2\Delta_1 \Delta_3 \delta_1 \delta_3 \cos(13) \\
 & - 2\Delta_1 \Delta_4 \delta_1 \delta_4 \cos(14) - 2\Delta_2 \Delta_3 \delta_2 \delta_3 \cos(23) \\
 & - 2\Delta_2 \Delta_4 \delta_2 \delta_4 \cos(24) - 2\Delta_3 \Delta_4 \delta_3 \delta_4 \cos(34) = 9V^2,
 \end{aligned}$$

where $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ are the areas of the faces opposite the points 1, 2, 3, 4 respectively, and $\cos(12)$ is the cosine of the angle between the faces Δ_1 and Δ_2 .

NOTE: In the expansion of the above determinant, we used the fact that

$$\begin{vmatrix} \lambda_{12} & \lambda_{23} & \lambda_{24} & 1 \\ \lambda_{13} & 0 & \lambda_{34} & 1 \\ \lambda_{14} & \lambda_{43} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -16\Delta_1 \Delta_2 \cos(12).$$

To prove this we choose a rectangular reference frame in the plane of the points 2, 3, 4, giving the points coordinates (x_2, y_2) , (x_3, y_3) , (x_4, y_4) respectively. The projection of the point 1 on this plane is given coordinates (x_1', y_1') . Then

$$\begin{vmatrix} x_2^2 + y_2^2 & -2x_2 & -2y_2 & 1 \\ x_3^2 + y_3^2 & -2x_3 & -2y_3 & 1 \\ x_4^2 + y_4^2 & -2x_4 & -2y_4 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & x_1' & y_1' & x_1'^2 + y_1'^2 \\ 1 & x_3 & y_3 & x_3^2 + y_3^2 \\ 1 & x_4 & y_4 & x_4^2 + y_4^2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -16\Delta_1 \Delta_2 \cos(12).$$

$$\text{Whence } \begin{vmatrix} \lambda_{1'2} & \lambda_{23} & \lambda_{24} & 1 \\ \lambda_{1'3} & 0 & \lambda_{34} & 1 \\ \lambda_{1'4} & \lambda_{43} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -16\Delta_1 \Delta_2 \cos(12).$$

Replacing in this determinant $\lambda_{1'i}$ by $\lambda_{1i} - p^2$, ($i = 2, 3, 4$), where p is the distance from the point 1 to the plane of 2, 3, 4, and then splitting the resulting determinant into sums of determinants, we have the result.

For given $\delta_2, \delta_3, \delta_4$ there are two values of δ_1 , or two planes, satisfying the equation (4.2). When will these planes coincide? Take three balls, no two of the same size, and place them on a table. Their centers will represent the points 2, 3, 4 and their radii $\delta_2, \delta_3, \delta_4$. The plane of the table will be one tangent plane and there will be also an upper external tangent plane, for which a piece of cardboard may be used. If we now bring the three points of contact of the spheres with the table into a line, the upper tangent plane will swing around and finally coincide with the table. Hence the three points

2, 3, 4 will be in a plane through the line of contact and perpendicular to the table. Then

$$(4.3) \quad a^2\delta_2^2 + b^2\delta_3^2 + c^2\delta_4^2 - 2ab \cos C \delta_2\delta_3 \\ - 2ac \cos B \delta_2\delta_4 - 2bc \cos A \delta_3\delta_4 = 4\Delta_1^2,$$

where a, b, c are the sides of the face Δ_1 opposite the points 2, 3, 4 respectively, and A, B, C are the interior angles at the points 2, 3, 4.

The same relation is true when the internal tangent planes coincide. Hence the discriminant of equation (4.2) with respect to δ_1 must reduce to equation (4.3) when the tangent planes coincide. The discriminant when set equal to zero, gives

$$\Delta_2^2 \sin^2 (12) \delta_2^2 + \Delta_3^2 \sin^2 (13) \delta_3^2 + \Delta_4^2 \sin^2 (14) \delta_4^2 \\ - 2\Delta_2\Delta_3\delta_2\delta_3 [\cos (23) + \cos (12) \cos (13)] \\ - 2\Delta_2\Delta_4\delta_2\delta_4 [\cos (24) + \cos (12) \cos (14)] \\ - 2\Delta_3\Delta_4\delta_3\delta_4 [\cos (34) + \cos (13) \cos (14)] = 9V^2.$$

This equation must reduce to equation (4.3). Therefore the following relations must hold true:

$$\Delta_2 \sin (12)/a = \Delta_3 \sin (13)/b = \Delta_4 \sin (14)/c, \\ \sin (12) \sin (13) \cos C = \cos (23) + \cos (12) \cos (13), \\ \sin (12) \sin (14) \cos B = \cos (24) + \cos (12) \cos (14), \\ \sin (13) \sin (14) \cos A = \cos (34) + \cos (13) \cos (14), \\ 2\Delta_1\Delta_2 \sin (12) = 3aV.*$$

Here we have information about the tetrahedron. The first set of equations gives the law of sines for the dihedral angles, the second set gives the law of cosines. The last equation is a formula for the volume.

This theory can be extended to any number of points. The doubly-bordered Cayley determinant for n points can always be set down. On expanding the determinant, we will always have a quadratic expression in the δ 's, such that the discriminant with respect to δ_1 , when set equal to zero, must reduce to the relation on the δ 's for $(n-1)$ points. The algebraic conditions imposed by this fact give information about the n point.

THE JOHNS HOPKINS UNIVERSITY.

* Casey, *Spherical Trigonometry*, p. 133.

VITA.

Joseph William Peters was born on November 30, 1903, at Baltimore, Md. He received his elementary education in the schools of Baltimore County and his high-school work was done at the Baltimore City College. In October, 1921, he entered the collegiate department of the Johns Hopkins University, from which he received his A. B. degree in 1925. Since that time he has been a graduate student and junior instructor in the department of mathematics in the University, receiving his M. A. degree in 1927.

